

# Universal dependence on disorder of two-dimensional randomly diluted and random-bond $\pm J$ Ising models

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We consider the two-dimensional randomly site diluted Ising model and the random-bond  $\pm J$  Ising model (also called the Edwards-Anderson model), and study their critical behavior at the paramagnetic-ferromagnetic transition. The critical behavior of thermodynamic quantities can be derived from a set of renormalization-group equations, in which disorder is a marginally irrelevant perturbation at the two-dimensional Ising fixed point. We discuss their solutions, focusing in particular on the universality of the logarithmic corrections arising from the presence of disorder. Then, we present a finite-size scaling analysis of high-statistics Monte Carlo simulations. The numerical results confirm the renormalization-group predictions, and in particular the universality of the logarithmic corrections to the Ising behavior due to quenched dilution.

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## I. INTRODUCTION AND SUMMARY

Random Ising systems represent an interesting theoretical laboratory in which one can study general features of disordered systems. Among them, the two-dimensional (2D) random-site and random-bond Ising models have attracted much interest. In particular, the effects of quenched disorder on the critical behavior at the paramagnetic-ferromagnetic transitions, which are observed for sufficiently small disorder, have been much investigated and debated; see Refs. [1–39].

Renormalization-group (RG) and conformal field theory [2,4,8,12] predict the marginal irrelevance of random dilution at the paramagnetic-ferromagnetic transition. Therefore, the asymptotic behavior is controlled by the standard Ising fixed point, characterized by the critical exponents  $\nu=1$  and  $\eta=\frac{1}{4}$ ; disorder gives rise only to logarithmic corrections. The marginality of quenched disorder coupled to the energy density, as is the case for random dilution, is already suggested by the Harris criterion [40], which states that the relevance or irrelevance of quenched dilution depends on the sign of the specific-heat exponent of the pure system; in the case of the 2D Ising model, the specific heat diverges only logarithmically at the transition, i.e.,  $\alpha=0^+$ . The marginal irrelevance of disorder has also been supported by numerical studies of lattice models; see, e.g., Refs. [10,23,24,26,28,32–34,36–38] (see, however, Refs. [13,14,30] for different scenarios). We recall that in three dimensions, random dilution is a relevant perturbation of the pure Ising fixed point, leading to a new three-dimensional randomly diluted Ising (RDI) universality class, which is characterized by different critical exponents; see, e.g., Refs. [41,42].

In this paper, we return to the issue of the critical behavior of 2D randomly diluted Ising systems. By using the RG results reported in Refs. [3,5,6,28], we show that their critical behavior can be derived from the RG equations,

$$\frac{du_l}{dl} = 2u_l + d_l u_l^2,$$

$$\frac{du_r}{dl} = u_r - \frac{1}{2} g u_r,$$

$$\frac{du_h}{dl} = \frac{15}{8} u_h,$$

$$\frac{dg}{dl} = -g^2 + \frac{1}{2} g^3, \quad (1)$$

where  $l$  is the flow parameter (the logarithm of a length scale);  $u_l$ ,  $u_r$ , and  $u_h$  are the scaling fields associated with the leading operators of the three different conformal families of the 2D Ising model, i.e., the identity, energy, and spin families; and  $g$  is the marginally irrelevant scaling field associated with disorder. Higher-order terms in Eqs. (1) are not necessary, because they can be reabsorbed by appropriate analytic redefinitions of the scaling fields. The appearance of the term  $d_l u_l^2$  in the first equation, where  $d_l$  is a nonuniversal constant, is due to the resonance of the identity operator with the thermal operator, which already occurs in the pure Ising model [43]. It is interesting to note that randomness couples only to the thermal scaling field  $u_r$ . It would be interesting to understand if these conclusions also apply to the irrelevant operators, i.e., if the only operators that couple disorder are those that belong to the conformal family of the energy.

The analysis of the RG equations shows that random dilution gives rise to logarithmic corrections that are universal after an appropriate normalization of the scaling field associated with disorder. Additional scaling corrections due to the irrelevant operators are suppressed by power laws as in standard continuous transitions. For these reasons, we prefer to distinguish the randomly dilute Ising (RDI) critical behavior characterized by the RG equations (1) from the standard 2D

Ising universality class of pure systems, although they share the same 2D Ising fixed point.

The RG equations (1) allow us to determine the scaling behavior of any thermodynamic quantity. Denoting with  $t$ ,  $h$ ,  $p$ , and  $L$  the reduced temperature, the magnetic external field, the disorder parameter, and the lattice size, respectively, the free energy satisfies the scaling equation

$$F(t, h, p, L) = e^{-2l} u_t(l) + e^{-2l} f(u_t(l), u_h(l), g(l), e^l L) \quad (2)$$

(we consider models defined on square  $L \times L$  lattices with periodic boundary conditions), where  $u_t(l)$ ,  $u_h(l)$ ,  $u_p(l)$ , and  $g(l)$  are the solutions of the RG equations. From Eq. (2), one can derive the scaling behavior of the relevant thermodynamic quantities and determine the logarithmic corrections due to the quenched disorder. At the critical point  $t=h=0$ , we obtain the asymptotic behaviors [28]

$$C_h \sim \ln \ln L \quad (3)$$

for the specific heat, and

$$\chi = cL^{7/4} f_\chi[g(\ln L)] = cL^{7/4} \left[ 1 + O\left(\frac{1}{\ln L}\right) \right], \quad (4)$$

$$R = R^* f_R[g(\ln L)] = R^* + O\left(\frac{1}{\ln L}\right), \quad (5)$$

$$\frac{dR}{dt} = cLg(\ln L)^{1/2} f_{dR}[g(\ln L)] = c \frac{L}{\sqrt{\ln L}} \left[ 1 + O\left(\frac{1}{\ln L}\right) \right], \quad (6)$$

for the magnetic susceptibility  $\chi$ , any RG invariant quantity  $R$ , such as the quartic Binder cumulant  $U_4$  and the ratio  $R_\xi \equiv \xi/L$ , and its derivative with respect to the temperature. Here  $R^*$  indicates the Ising fixed-point value and  $g(\ln L)$  is the solution of Eq. (1) with  $l = \ln L$ . For  $L \rightarrow \infty$ ,  $g(\ln L)$  behaves as

$$g(\ln L) \sim \frac{1}{\ln L/L_0} \left( 1 + \frac{\ln \ln L/L_0}{2 \ln L/L_0} + \dots \right), \quad (7)$$

where  $L_0$  is a length scale. The functions  $f_\chi(x)$ ,  $f_R(x)$ , and  $f_{dR}(x)$  are normalized such that  $f_\#(0)=1$  and are universal once the scaling field  $g(\ln L)$  is appropriately normalized. In the above-reported equations, we have neglected scaling corrections that are suppressed by power laws. They are due to the analytic dependence of the scaling fields on the Hamiltonian parameters, to the background (i.e., the contribution of the identity operator in the RG language), and to the irrelevant operators [44,45]. In particular, we expect scaling corrections associated with the leading irrelevant operator appearing in the pure Ising model (the corresponding exponent is  $\omega=2$ ).

Moreover, in this paper we compare the theoretical predictions with a finite-size scaling (FSS) analysis of numerical Monte Carlo (MC) results for the randomly site-diluted Ising model and for the random-bond  $\pm J$  Ising model, also known as the Edwards-Anderson model. Our main results can be summarized as follows. Our FSS analyses provide robust evidence that the paramagnetic-ferromagnetic transitions in

these models present the same RDI critical behavior. Note that this implies that frustration in the random-bond  $\pm J$  Ising model is irrelevant at the paramagnetic-ferromagnetic transition line. The FSS behaviors are in agreement with the predictions of the RG equations (1). The asymptotic critical behavior appears to be controlled by the Ising fixed point. The logarithmic corrections are consistent with the existence of universal scaling functions as predicted by Eqs. (1); cf. Eqs. (2)–(4), (6), and (7).

The paper is organized as follows. In Sec. II, we define the randomly site-diluted model and the  $\pm J$  Ising model, we briefly discuss their phase diagrams, and we define the quantities that we consider in the paper. In Sec. III, we discuss the RG flow at a 2D Ising fixed point in the presence of a marginally irrelevant operator, and the implications for the infinite-volume and finite-size critical behavior. In particular, we focus on the universal features of the logarithmic corrections due to disorder. Finally, in Sec. IV we present our FSS analysis of high-statistics MC results for the randomly site-diluted and random-bond  $\pm J$  Ising models.

## II. RANDOMLY SITE-DILUTED AND RANDOM-BOND $\pm J$ ISING MODELS

### A. The models and their phase diagrams

The randomly site-diluted Ising model (RSIM) is defined by the Hamiltonian

$$\mathcal{H}_s = - \sum_{\langle xy \rangle} \rho_x \rho_y \sigma_x \sigma_y, \quad (8)$$

where the sum is extended over all pairs of nearest-neighbor sites of a square lattice,  $\sigma_x = \pm 1$  are Ising spin variables, and  $\rho_x$  are uncorrelated quenched random variables, which are equal to 1 with probability  $p$  (the spin concentration) and zero with probability  $1-p$  (the impurity concentration). The RSIM is expected to undergo a continuous transition for any  $p > p_{\text{perc}}$ , where [46]  $p_{\text{perc}} = 0.593\,746\,21(13)$  corresponds to the site-percolation point of the impurities; moreover,  $T_c \rightarrow 0$  for  $p \rightarrow p_{\text{perc}}$ ; see, e.g., Ref. [47]. For  $p \leq p_{\text{perc}}$ , the ferromagnetic phase is absent. Thus, the paramagnetic-ferromagnetic transition line starts from the pure Ising point  $X_{\text{Is}} = (p=1, T=T_{\text{Is}})$ , where  $T_{\text{Is}} = 2/\ln(1+\sqrt{2}) = 2.269\,19\dots$  is the critical temperature of the 2D Ising model, and ends at  $X_{\text{perc}} = (p=p_{\text{perc}}, T=0)$ . Along this line, the critical behavior is expected to be universal, i.e., independent of dilution, and to be characterized by the RG equations (1). As we shall see, this is supported by the analysis of our MC results.

The random-bond  $\pm J$  Ising model, also known as the Edwards-Anderson model [48], is defined by the lattice Hamiltonian

$$\mathcal{H}_b = - \sum_{\langle xy \rangle} J_{xy} \sigma_x \sigma_y, \quad (9)$$

where  $\sigma_x = \pm 1$ , the sum is over all pairs of nearest-neighbor sites of a square lattice, and the exchange interactions  $J_{xy}$  are uncorrelated quenched random variables, taking values  $\pm J$  with probability distribution

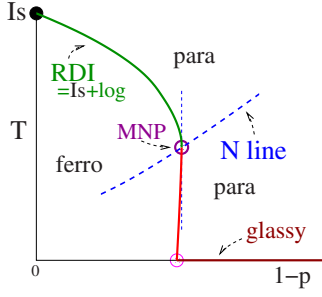


FIG. 1. (Color online) Phase diagram of the square-lattice random-bond  $\pm J$  Ising (Edwards-Anderson) model in the  $T$ - $p$  plane.

$$P(J_{xy}) = p\delta(J_{xy} - J) + (1-p)\delta(J_{xy} + J). \quad (10)$$

In the following, we set  $J=1$  without loss of generality. For  $p=1$ , we recover the standard Ising model, while for  $p=\frac{1}{2}$  we obtain the bimodal Ising spin-glass model. The  $\pm J$  Ising model is a simplified model [48] for disordered spin systems showing glassy behavior in some region of their phase diagram. Its phase diagram in two dimensions is sketched in Fig. 1 (it is symmetric for  $p \rightarrow 1-p$ ). For sufficiently small values of the probability  $p_a \equiv 1-p$  of the antiferromagnetic bonds, the model presents a paramagnetic phase and a ferromagnetic phase. The paramagnetic-ferromagnetic transition line starts from the Ising point  $X_{Is}=(p=1, T=T_{Is})$  and extends up to the multicritical Nishimori point (MNP)  $X^*=(p^*, T^*)$ , located along the so-called Nishimori line ( $N$  line) defined by  $2p-1=\tanh(1/T)$  [49–52], with [53]  $p^*=0.89081(7)$  and  $T^*=0.9528(4)$ . The critical behavior is expected to be in the same universality class as that of the transition in the RSIM. As we shall see, our FSS analysis strongly supports this scenario. A detailed discussion of the phase diagram can be found in Ref. [53] and references therein.

### B. Observables

In our FSS analyses, we consider models defined on a square  $L \times L$  lattice with periodic boundary conditions. The two-point correlation function is defined as

$$G(x) \equiv [\langle \sigma_0 \sigma_x \rangle], \quad (11)$$

where the angular and square brackets indicate the thermal average and the quenched average over disorder, i.e., over  $\rho_x$  in the case of RSIM and over  $J_{xy}$  in the case of the  $\pm J$  Ising model. We define the magnetic susceptibility  $\chi \equiv \sum_x G(x)$  and the correlation length  $\xi$ ,

$$\xi^2 \equiv \frac{\tilde{G}(0) - \tilde{G}(q_{\min})}{\hat{q}_{\min}^2 \tilde{G}(q_{\min})}, \quad (12)$$

where  $q_{\min} \equiv (2\pi/L, 0)$ ,  $\hat{q} \equiv 2 \sin q/2$ , and  $\tilde{G}(q)$  is the Fourier transform of  $G(x)$ . We also consider quantities that are invariant under RG transformations in the critical limit. Beside the ratio

$$R_\xi \equiv \xi/L, \quad (13)$$

we consider the quartic cumulants  $U_4$  and  $U_{22}$  defined by

$$U_4 \equiv \frac{[\mu_4]}{[\mu_2]^2}, \quad U_{22} \equiv \frac{[\mu_2^2] - [\mu_2]^2}{[\mu_2]^2}, \quad (14)$$

where

$$\mu_k \equiv \left\langle \left( \sum_x \sigma_x \right)^k \right\rangle. \quad (15)$$

The above RG-invariant quantities  $R_\xi$ ,  $U_4$ , and  $U_{22}$  are also called phenomenological couplings. In the critical ( $T=T_c$ ) 2D pure Ising model, they converge for large  $L$  to the universal values [54]

$$R_\xi^* = R_{Is} = 0.9050488292(4), \quad (16)$$

$$U_4^* = U_{Is} = 1.167923(5), \quad (17)$$

$$U_{22}^* = 0. \quad (18)$$

Finally, we consider the derivatives

$$R'_\xi \equiv \frac{dR_\xi}{d\beta}, \quad U'_4 \equiv \frac{dU_4}{d\beta}, \quad (19)$$

which can be computed by measuring appropriate expectation values at fixed  $\beta$  and  $p$ .

## III. RENORMALIZATION-GROUP FLOW AND FINITE-SIZE SCALING

### A. Ising RG flow in the presence of a marginally irrelevant scaling field associated with disorder

In this section, we discuss the RG flow close to the 2D Ising fixed point in the presence of a marginally irrelevant scaling field associated with disorder.

Let us consider a system with a marginal scaling field  $\hat{u}_0 \equiv \hat{g}$  and with a set of scaling fields  $\hat{u}_k$ ,  $k \geq 1$ , with RG dimensions  $y_k \neq 0$ . Close to the fixed point  $\hat{g} = \hat{u}_1 = \dots = 0$ , the RG equations have the form

$$\begin{aligned} \frac{d\hat{g}}{dl} &= \sum_{0 \leq i \leq j} b_{0,ij} \hat{u}_i \hat{u}_j + \sum_{0 \leq i \leq j \leq m} b_{0,ijm} \hat{u}_i \hat{u}_j \hat{u}_m + \dots, \\ \frac{d\hat{u}_k}{dl} &= y_k \hat{u}_k + \sum_{0 \leq i \leq j} b_{k,ij} \hat{u}_i \hat{u}_j + \sum_{0 \leq i \leq j \leq m} b_{k,ijm} \hat{u}_i \hat{u}_j \hat{u}_m + \dots. \end{aligned} \quad (20)$$

If there are no degeneracies ( $y_k \neq y_h$  for all  $k \neq h$ ) and no resonancies (i.e., there is no combination with integer coefficients of the RG dimensions that vanishes), one can redefine the scaling fields in such a way to simplify the RG equations. We define

$$g = \hat{g} + \sum_{0 \leq i \leq j} c_{0,ij} \hat{u}_i \hat{u}_j + \sum_{0 \leq i \leq j \leq m} c_{0,ijm} \hat{u}_i \hat{u}_j \hat{u}_m + \dots, \quad (21)$$

$$u_k = \hat{u}_k + \sum_{0 \leq i \leq j} c_{k,ij} \hat{u}_i \hat{u}_j + \sum_{0 \leq i \leq j \leq m} c_{k,ijm} \hat{u}_i \hat{u}_j \hat{u}_m + \dots \quad (22)$$

With a proper choice of the coefficients  $c_{k,ij}$ ,  $c_{k,ijm}, \dots$ , we can simplify the RG equations, obtaining the simple canonical form

$$\frac{dg}{dl} = -b_2 g^2 - b_3 g^3, \quad (23)$$

$$\frac{du_k}{dl} = y_k u_k + c_k g u_k. \quad (24)$$

By normalizing appropriately the scaling field  $g$ , we can also set  $|b_2|=1$ . In the case we are considering,  $g$  is marginally irrelevant so that  $b_2 > 0$  [we assume that  $g(l=0) > 0$ ]. We can thus set  $b_2 = 1$ . Once this choice has been made,  $b_3$  and all coefficients  $c_k$  are universal.

The simple form we have derived above does not strictly apply to the RSIM. Indeed, in the Ising model the RG operators belong to three different conformal families and within each family the RG dimensions differ by integers (see Ref. [45] for a discussion of the irrelevant operators in the pure Ising model). Thus, in the present case there are both degeneracies and resonancies. If we limit our considerations to the relevant scaling fields, we must only consider the resonance between the identity operator and the thermal operator, which is responsible for the logarithmic divergence of the specific heat in the pure Ising model [43]. By a proper redefinition of the nonlinear scaling fields, one can show that in this case the RG equations (20) for the relevant scaling fields can be written as

$$\frac{du_t}{dl} = 2u_t + c_t g u_t + d_t u_t^2, \quad (25)$$

$$\frac{du_l}{dl} = y_l u_l + c_l g u_l, \quad (26)$$

$$\frac{du_h}{dl} = y_h u_h + c_h g u_h, \quad (27)$$

$$\frac{dg}{dl} = -g^2 - b_3 g^3, \quad (28)$$

where the couplings to the irrelevant scaling fields due to the additional resonancies have been neglected. The scaling field  $u_l$  is associated with the identity operator. The additional term  $d_t u_t^2$ , which appears in Eq. (25), is due to the resonance with the thermal operator, as in the pure Ising model [43]. The scaling fields  $u_t$  and  $u_h$  are the relevant scaling fields associated with the reduced temperature  $t$  and the external field  $h$ , respectively;  $y_t = 1$  and  $y_h = 15/8$  are the corresponding RG dimensions. Finally,  $g$  is the marginally irrelevant operator associated with randomness. The coefficients  $c_t$ ,  $c_l$ ,  $c_h$ , and  $b_3$  are universal, being independent of the normalization of the scaling fields. By using conformal field theory,  $c_t$ ,  $c_l$ ,  $c_h$ , and  $b_3$  have been computed [3,5,6,19,28],

$$c_t = -\frac{1}{2}, \quad c_h = 0, \quad b_3 = -\frac{1}{2}. \quad (29)$$

Let us now integrate the RG equations. Since  $b_3 < 0$ , Eq. (28) has two fixed points with  $g \geq 0$ : one is  $g=0$  and is stable; the second one is  $g = -1/b_3 = 2$  and is unstable. Thus, the basin of attraction of the Ising FP corresponds to  $g_0 = g(l=0) < -1/b_3 = 2$ ; for  $g_0 > 2$ ,  $g(l)$  flows to infinity. It is important to note that Eq. (28) is only valid within the basin of attraction of the stable fixed point  $g=0$ . The redefinitions of the scaling fields that we have used to obtain the simple canonical form (28) cannot be extended outside the basin of attraction since they are expected to become singular at the unstable fixed point. The presence of an unstable fixed point indicates that the behavior for large values of the disorder is not controlled by the Ising fixed point. The RG flow could be attracted by a new fixed point—thus, for large values of the disorder the transition would be continuous and in a new universality class—or could go to infinity, indicating the absence of a continuous transition. A similar phenomenon was conjectured in three dimensions [55] on the basis of a perturbative field-theoretical analysis of the RG flow.

If  $g_0 < -1/b_3$ , the function  $g(l)$  is implicitly given by (we do not replace  $b_3$  with its theoretical value  $-1/2$ , in order to obtain general expressions that can be tested numerically)

$$F[g(l)] = l + F(g_0),$$

$$F(x) \equiv \frac{1}{x} + b_3 \ln \left( \frac{x}{1 + b_3 x} \right). \quad (30)$$

The solution can be simplified if we introduce

$$\tilde{g}(l) = \frac{g(l)}{1 + b_3 g(l)}, \quad (31)$$

which satisfies the implicit equation

$$\tilde{F}[\tilde{g}(l)] = l + \tilde{F}(\tilde{g}_0),$$

$$\tilde{F}(x) \equiv \frac{1}{x} + b_3 \ln x. \quad (32)$$

This equation can be inverted to give

$$\tilde{g}(l) = \Phi(\tilde{g}_0, l). \quad (33)$$

The function  $\Phi(x, l)$  cannot be computed analytically. However, it is easy to determine it in the large- $l$  limit. We obtain

$$\tilde{g}(l) = \frac{1}{y} - \frac{b_3 \ln y}{y^2} + \frac{b_3^2 (\ln^2 y - \ln y)}{y^3} + O\left(\frac{b_3^3 \ln^3 y}{y^4}\right), \quad (34)$$

where  $y \equiv l + \tilde{F}(\tilde{g}_0)$ . Since  $c_h = 0$ , the equation for  $u_h$  gives

$$u_h(l) = u_{h,0} e^{y_h l}, \quad (35)$$

where  $u_{h,0} = u_h(l=0)$ . In order to determine  $u_t(l)$ , we rewrite the corresponding equation as



$$\frac{du_t}{u_t} = y_t dl - c_t \frac{g dg}{g^2 + b_3 g^3} = y_t dl - c_t \frac{d\tilde{g}}{\tilde{g}}, \quad (36)$$

which gives ( $c_t = -1/2$ ,  $y_t = 1$ )

$$u_t(l) = u_{t,0} e^{l \left[ \frac{\tilde{g}(l)}{\tilde{g}_0} \right]^{1/2}}, \quad (37)$$

where  $u_{t,0} = u_t(l=0)$ . For large  $l$ , the function  $u_t(l)$  behaves as

$$u_t(l) = u_{t,0} \tilde{g}_0^{-1/2} \frac{e^l}{l^{1/2}}. \quad (38)$$

Let us finally consider the identity operator. If  $d_I = 0$ , the solution can be obtained as in the case of  $u_t$ . Thus, we write

$$u_I(l) = u_{I,0} e^{2l} K(l) \left[ \frac{\tilde{g}(l)}{\tilde{g}_0} \right]^{-c_I}, \quad (39)$$

where  $K(l)$  is an unknown function of  $l$ , such that  $K(l=0) = 1$ . Substituting in the equation for  $u_I(l)$  and using the result for  $u_t(l)$ , we obtain

$$\frac{dK}{dl} = d_I \frac{u_{t,0}^2}{u_{I,0}} \left[ \frac{\tilde{g}(l)}{\tilde{g}_0} \right]^{c_I+1}, \quad (40)$$

and therefore

$$K(l) = 1 - d_I \frac{u_{t,0}^2}{u_{I,0}} \tilde{g}_0^{-c_I} \int_{\tilde{g}_0}^{\tilde{g}(l)} dx x^{c_I-1} (1 - b_3 x). \quad (41)$$

The behavior of  $u_I(l)$  for  $l \rightarrow \infty$  depends on the value of  $c_I$ . Since the integral appearing in  $K(l)$  diverges as  $l^{-c_I}$  for  $c_I < 0$ , as  $\ln l$  for  $c_I = 0$ , and is finite for  $c_I > 0$ , we obtain

$$u_I(l) \sim \begin{cases} e^{2l} l^{c_I} & \text{for } c_I > 0, \\ e^{2l} \ln l & \text{for } c_I = 0, \\ e^{2l} & \text{for } c_I < 0. \end{cases} \quad (42)$$

The RG equations do not fix completely the normalization of the scaling fields. First, one can redefine  $u_t$ ,  $u_h$ , and  $u_I$  by a multiplicative constant [56]; such a redefinition is not possible for  $g$ , since a multiplicative constant would break the condition  $b_2 = 1$ . Beside these trivial redefinitions, there is also a nonlinear set of transformations that leave the equations invariant. Given a constant  $\lambda$ , we define  $g_\lambda$  as the solution of the equation

$$F(g_\lambda) = F(g) + \lambda. \quad (43)$$

Then, for any  $\lambda$  we have

$$\frac{dg_\lambda}{dl} = -g_\lambda^2 - b_3 g_\lambda^3. \quad (44)$$

Note that the transformation is analytic in a neighborhood of  $g=0$ . If  $\tilde{g}_\lambda$  is defined as in Eq. (31), we obtain

$$\tilde{g}_\lambda = \tilde{g} [1 - \lambda \tilde{g} + O(\tilde{g}^2)]. \quad (45)$$

Analogously, if we define

$$u_{t,\lambda} = u_t (\tilde{g}/\tilde{g}_\lambda)^{-1/2}, \quad (46)$$

then  $u_{t,\lambda}$  satisfies the same equation of  $u_t$  with  $g_\lambda$  replacing  $g$ , as can be seen from Eq. (37). A similar redefinition can be

made for  $u_I$ . This invariance implies that, beside fixing the normalizations of  $u_t$ ,  $u_h$ , and  $u_I$ , we must also appropriately fix  $g$ . In practical terms,  $F(g_0)$  is completely arbitrary and must be fixed in order to define  $g(l)$  unambiguously. Finally, note that there are no analytic redefinitions of  $g$  that map Eq. (28) in an identical equation with  $b'_3 \neq b_3$ , proving the universality of  $b_3$ .

Neglecting scaling corrections that are suppressed by power laws, we write the free energy in the scaling form [43]

$$\mathcal{F}(t, h, p) = e^{-2l} u_t(l) + e^{-2l} f_{\text{sing}}(u_t(l), u_h(l), g(l)) \quad (47)$$

for any  $l$ . Note that the whole dependence on  $t \equiv T/T_c - 1$ ,  $h$ , and  $p$  is encoded in the constants  $\tilde{g}_0$ ,  $u_{t,0}$ ,  $u_{h,0}$ ,  $u_{I,0}$ , and  $d_I$ . Of course,  $u_{t,0} \sim t$  and  $u_{h,0} \sim h$  vanish at the critical point, while  $\tilde{g}_0$  vanishes for  $p=1$ . The independence of Eq. (47) on  $l$  allows us to simplify the general expression for the free energy. We choose  $l$  such that  $u_t(l) = 1$  and thus

$$e^l = \tau^{-1} (-\ln \tau)^{1/2} \left[ 1 + O\left( \frac{\ln |\ln \tau|}{\ln \tau} \right) \right],$$

$$\tilde{g}(l) = -\frac{1}{\ln \tau} \left[ 1 + O\left( \frac{\ln |\ln \tau|}{\ln \tau} \right) \right], \quad (48)$$

where  $\tau = u_{t,0}/\tilde{g}_0^{1/2}$ . Substituting these expressions in Eq. (47), we obtain the general dependence of the free energy on  $t$  and  $h$ .

In order to determine  $c_I$ , we consider the specific heat  $C_h$ . The leading singular behavior is due to the temperature dependence of the scaling field  $u_I$ . Using Eq. (42), we obtain

$$C_h \propto \frac{\partial^2 \mathcal{F}(t, 0, p)}{\partial t^2} \sim \begin{cases} (\ln 1/t)^{c_I} & \text{for } c_I > 0, \\ \ln \ln(1/t) & \text{for } c_I = 0, \\ \text{constant} & \text{for } c_I < 0. \end{cases} \quad (49)$$

The asymptotic behavior of the specific heat of 2D randomly diluted Ising systems has been determined in Refs. [1,2,4], obtaining

$$C_h \sim \ln \ln(1/t). \quad (50)$$

Comparing with Eq. (49), we obtain  $c_I = 0$ . In this case, we have

$$u_I(l) = u_{I,0} e^{2l} - \frac{d_I u_{t,0}^2 e^{2l}}{\tilde{g}_0} \left[ \ln \frac{\tilde{g}(l)}{\tilde{g}_0} - b_3 (\tilde{g} - \tilde{g}_0) \right]. \quad (51)$$

It is interesting to note that, since  $c_h = c_I = 0$ , randomness couples only to the thermal scaling field  $u_t$ . This result appears quite natural from the point of view of the Landau-Ginzburg-Wilson approach to critical phenomena. Indeed, in field theory, randomly diluted models are obtained by coupling disorder to the energy operator [57,58],

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} [\partial_\mu \phi(x)]^2 + \frac{1}{2} r \phi(x)^2 + \frac{1}{2} \psi(x) \phi(x)^2 + \frac{1}{4!} g_0 [\phi(x)^2]^2 \right\}, \quad (52)$$

where  $r \propto T - T_c$ , and  $\psi(x)$  is a spatially uncorrelated random field with Gaussian distribution. The 2D RDI critical behav-

ior has been also investigated by using this field-theoretical approach and the so-called replica trick. The analysis of the corresponding five-loop perturbative expansions [7,34] gives results that are substantially consistent with the marginal irrelevance of disorder.

### B. Finite-size scaling

Let us now discuss the implications of the above RG analysis for the FSS of thermodynamic quantities at the critical point. We start from the scaling behavior of the free energy,

$$\mathcal{F}(t, h, p, L) = e^{-2l} u_l(l) + e^{-2l} f(u_l(l), u_h(l), g(l), e^l L^{-1}), \quad (53)$$

where the contributions of the irrelevant scaling fields have been neglected. By choosing  $l = \ln L$ , we can write

$$\mathcal{F}(t, h, p, L) = L^{-2} u_l(\ln L) + L^{-2} f(u_l(\ln L), u_h(\ln L), g(\ln L)). \quad (54)$$

If we set  $\tilde{F}(\tilde{g}_0) = -\ln L_0$  in Eq. (32), for  $L \rightarrow \infty$  we have

$$\tilde{g}(\ln L) = \frac{1}{\ln L/L_0} \left[ 1 - b_3 \frac{\ln \ln L/L_0}{\ln L/L_0} + O\left(\frac{1}{(\ln L/L_0)^2}\right) \right]. \quad (55)$$

The free energy can then be written as

$$\mathcal{F}(t, h, p, L) = k_1 \ln \tilde{g}(\ln L) + k_2 + k_3 \tilde{g}(\ln L) + f(u_{t,0} L \tilde{g}(\ln L)^{1/2}, u_{h,0} L^{15/8}, \tilde{g}(\ln L)). \quad (56)$$

The constants  $k_i$ ,  $u_{t,0}$ ,  $u_{h,0}$ , and  $L_0$  depend on  $t$ ,  $h$ , and  $p$ . Moreover,  $u_{t,0} \sim t$  and  $u_{h,0} \sim u_h \sim h$  close to the critical point. The terms proportional to  $k_1$ ,  $k_2$ , and  $k_3$  are due to the identity operator, whose dependence on  $\tilde{g}(\ln L)$  is given in Eq. (51). Equation (56) is valid up to contributions of the irrelevant operators, which are expected to scale as inverse powers of  $L$ .

From Eq. (56), we can compute zero-momentum quantities that involve disorder averages of a single thermal average. For instance, for the specific heat at  $T = T_c$  and  $h = 0$  we obtain

$$C_h \sim \ln \ln L. \quad (57)$$

For the susceptibility at  $h = 0$ , we obtain

$$\chi = \left( \frac{\partial u_{h,0}}{\partial h} \right)^2 L^{7/4} f_\chi(u_{t,0} L \tilde{g}(\ln L)^{1/2}, \tilde{g}(\ln L)), \quad (58)$$

where, as before, we neglect power-law scaling corrections. A similar scaling equation holds for  $U_4$ ,

$$U_4 = f_{U_4}(u_{t,0} L \tilde{g}(\ln L)^{1/2}, \tilde{g}(\ln L)). \quad (59)$$

The determination of the scaling behavior of  $U_{22}$  and  $R_\xi \equiv \xi/L$  requires an extension of the scaling ansatz (56). A detailed discussion is presented in Sec. 3.1 of Ref. [42]. It shows that both quantities behave as  $U_4$ , apart from scaling corrections. Thus, if  $R = U_{22}$  or  $R = R_\xi$ , we also have

$$R = f_R(u_{t,0} L \tilde{g}(\ln L)^{1/2}, \tilde{g}(\ln L)). \quad (60)$$

Derivatives of the phenomenological couplings have a simple behavior as well, the leading term being of the form

$$\frac{\partial R}{\partial \beta} = \left( \frac{\partial u_{t,0}}{\partial t} \right) L \tilde{g}(\ln L)^{1/2} f_{dR}(u_{t,0} L \tilde{g}(\ln L)^{1/2}, \tilde{g}(\ln L)). \quad (61)$$

At the critical point we can set  $u_{t,0} = 0$ , so that we can write the scaling behaviors

$$R = g_R[\tilde{g}(\ln L)],$$

$$\frac{\partial R}{\partial \beta} = \left( \frac{\partial u_{t,0}}{\partial t} \right) L \tilde{g}(\ln L)^{1/2} g_{dR}[\tilde{g}(\ln L)]. \quad (62)$$

The functions  $g_R(x)$  and  $g_{dR}(x)$  are universal once an appropriate normalization is chosen for  $\tilde{g}(\ln L)$ , which is independent of the model. For this purpose, let us consider a phenomenological coupling  $R$ . For  $L \rightarrow \infty$ , we can expand

$$R = R^* + r_1 \tilde{g}(\ln L) + r_2 \tilde{g}(\ln L)^2 + \dots \quad (63)$$

Now we normalize  $\tilde{g}(\ln L)$  by requiring  $r_2 = 0$ . It is easy to prove that this is a correct normalization condition. Indeed, imagine that  $\tilde{g}(\ln L)$  has been normalized arbitrarily so that  $r_2 \neq 0$ . Then, redefine  $\tilde{g}(\ln L)$  by using Eq. (45). By properly choosing  $\lambda$ , it is easy to see that one can set  $r_2 = 0$ . This condition fixes uniquely the scale  $L_0$ .

Note that, in the pure Ising model, we have  $U_{22} = 0$ , so that we expect at the critical point

$$U_{22} \sim \tilde{g}(\ln L) \quad (64)$$

for  $L \rightarrow \infty$ . It is natural to invert this relation to express  $\tilde{g}(\ln L)$  in terms of  $U_{22}(L)$ . Then, we obtain the scaling forms

$$R(L) = \tilde{f}_R(U_{22}), \quad (65)$$

$$\chi(L) = d_\chi L^{7/4} \tilde{f}_\chi(U_{22}), \quad (66)$$

$$\frac{\partial R(L)}{\partial \beta} = d_{dR} L U_{22}^{1/2} \tilde{f}_{dR}(U_{22}), \quad (67)$$

where  $\tilde{f}_R(x)$ ,  $\tilde{f}_\chi(x)$ , and  $\tilde{f}_{dR}(x)$  are universal scaling functions that are normalized such that  $\tilde{f}_R(0) = R^*$ ,  $\tilde{f}_\chi(0) = \tilde{f}_{dR}(0) = 1$  and have a regular expansion in powers of  $x$ . Note that these scaling equations are much simpler than those in terms of  $\tilde{g}(\ln L)$ , since they are independent of the scale  $L_0$  and of the normalization of  $\tilde{g}(\ln L)$ .

### C. Finite-size scaling at a fixed phenomenological coupling

Instead of computing the various quantities at fixed Hamiltonian parameters, one may study FSS keeping a phenomenological coupling  $R$  fixed at a given value  $R_f$ , as proposed in Ref. [59] and also discussed in Refs. [42,60]. This means that, for each  $L$ , one considers  $\beta_f(L)$  such that

$$R(\beta = \beta_f(L), L) = R_f. \quad (68)$$

All interesting thermodynamic quantities are then computed at  $\beta = \beta_f(L)$ . The pseudocritical temperature  $\beta_f(L)$  converges to  $\beta_c$  as  $L \rightarrow \infty$ .

In the next section, we report a FSS analysis of MC simulations keeping the phenomenological coupling  $R_\xi$  fixed. The value  $R_f$  can be specified at will as long as it is between the corresponding high-temperature and low-temperature values. Since we wish to check the hypothesis that the asymptotic critical behavior is governed by the Ising fixed point, we choose  $R_{\xi_f} = R_{\text{Is}}$ , where  $R_{\text{Is}} = 0.905\,048\,829\,2(4)$  is the universal value of  $R_\xi \equiv \xi/L$  at the critical point in the 2D Ising universality class [54] for square  $L \times L$  lattices with periodic boundary conditions. Note, however, that this choice does not bias our analysis in favor of the Ising nature of the transition. By fixing  $R_\xi$  to the critical Ising value, we can perform the following consistency check: if the transition belongs to the Ising universality class, then any other RG-invariant quantity must converge to its critical-point value in the Ising model.

In the  $(t, L)$  plane, the line  $R_\xi = R_{\text{Is}}$  is obtained by solving the equation

$$f_{R_\xi}(u_{t,0} L \tilde{g}(\ln L)^{1/2}, \tilde{g}(\ln L)) = R_{\text{Is}}. \quad (69)$$

It gives a relation

$$u_{t,0} L \tilde{g}(\ln L)^{1/2} = k[\tilde{g}(\ln L)], \quad (70)$$

where  $k(x)$  has a regular expansion in powers of  $x$ . Moreover, since we have chosen  $R_{\xi_f} = R_{\text{Is}}$ , we have  $k(0) = 0$ . Substituting relation (70) in the above-reported scaling equations for the susceptibility  $\chi$ , the phenomenological couplings  $R$ , and their derivative, we obtain at fixed  $R_\xi$

$$\chi(L) = c_\chi L^{7/4} C_\chi[\tilde{g}(\ln L)], \quad (71)$$

$$R(L) = C_R[\tilde{g}(\ln L)], \quad (72)$$

$$\frac{dR(L)}{d\beta} = c_{dR} L \tilde{g}(\ln L)^{1/2} C_{dR}[\tilde{g}(\ln L)]. \quad (73)$$

The scaling functions are universal, have a regular expansion in powers of  $\tilde{g}(\ln L)$ , and are normalized such that  $C_R(0) = R_{\text{Is}}$ ,  $C_\chi(0) = C_{dR}(0) = 1$ . The additional corrections due to the irrelevant operators decay as powers of  $1/L$ .

The large- $L$  behavior of  $\beta_f(L)$  follows from Eq. (70). Since  $k(x) \sim x + O(x^2)$ , we obtain

$$\begin{aligned} \beta_f - \beta_c &= \frac{c_1 \tilde{g}(\ln L)^{1/2}}{L} \\ &= \frac{c_1}{L \sqrt{\ln L/L_0}} \left[ 1 - \frac{b_3 \ln \ln L/L_0}{2 \ln L/L_0} + O\left(\frac{1}{\ln L/L_0}\right) \right], \end{aligned} \quad (74)$$

where  $L_0$  is computed at the critical point  $t=0$ .

We finally mention that Eqs. (65)–(67) hold at fixed  $R_\xi = R_{\text{Is}}$  as well. The corresponding universal scaling functions depend on the values of  $U_{22}$  at  $R_\xi = R_{\text{Is}}$  fixed, i.e.,  $\bar{U}_{22}(L)$

$= U_{22}(\beta_f(L), L)$  [we denote them by  $\bar{f}_R(\bar{U}_{22})$ ,  $\bar{f}_\chi(\bar{U}_{22})$ , and  $\bar{f}_{dR}(\bar{U}_{22})$ , respectively] and have a regular expansion in powers of  $\bar{U}_{22}$ .

## IV. FINITE-SIZE SCALING ANALYSIS OF MONTE CARLO SIMULATIONS

### A. Monte Carlo simulations

We perform high-statistics MC simulations of the RSIM at  $p=0.9, 0.7$ , and of the  $\pm J$  Ising model at  $p=0.95$ . We consider square lattices of linear size  $L$  with periodic boundary conditions. In the MC simulations of the RSIM, we use a mixture of Metropolis and Wolff cluster [61] updates as we did in the three-dimensional case reported in Ref. [42]. In the case of the  $\pm J$  Ising model, the Wolff cluster update is expected to be slow [62] so that we only use Metropolis updates with multispin coding.

Instead of computing the different quantities at fixed Hamiltonian parameters, we compute them at fixed  $R_\xi \equiv \xi/L = R_{\text{Is}}$ . This means that, given a MC sample generated at  $\beta = \beta_{\text{run}}$ , we determine the value  $\beta_f$  such that  $R_\xi(\beta = \beta_f) = R_f$ . All interesting observables are then computed at  $\beta = \beta_f$ . The pseudocritical temperature  $\beta_f$  converges to  $\beta_c$  as  $L \rightarrow \infty$ . This method has the advantage that it does not require a precise knowledge of the critical value  $\beta_c$  (an estimate is only needed to fix  $\beta_{\text{run}}$  that should be close to  $\beta_c$ ). Moreover, for some observables the statistical errors at fixed  $R_\xi$  are smaller than those at fixed  $\beta$  (close to  $\beta_c$ ) [42,60]. In order to compute any quantity at  $\beta = \beta_f$ , we determine its Taylor expansion around  $\beta_{\text{run}}$ , as we did in our previous work [62]. Particular care has been taken to avoid any bias due to the finite number of iterations for each sample: we use the method described in Ref. [42] and extended to correlated data in Ref. [62].

The results at fixed  $R_\xi = R_{\text{Is}}$  are reported in Table I. For each model and lattice size  $L$ , we report the number  $N_s$  of samples, the MC estimates of the quartic cumulants  $U_4$  and  $U_{22}$  at fixed  $R_\xi = R_{\text{Is}}$  (we denote them with  $\bar{U}_4$  and  $\bar{U}_{22}$ , respectively), the magnetic susceptibility  $\chi$ , the derivative  $R'_\xi \equiv dR_\xi/d\beta$ , and the specific heat  $C_h$ .

### B. Results

#### 1. Approach to the 2D Ising fixed-point values

Since we perform our FSS analysis keeping  $R_\xi = R_{\text{Is}}$  fixed, if the critical behavior is controlled by the Ising fixed point, in the large- $L$  limit we should have

$$\bar{U}_{22}(L) \rightarrow 0, \quad \bar{U}_4(L) \rightarrow U_{\text{Is}}, \quad (75)$$

where [54]  $U_{\text{Is}} = 1.167\,923(5)$  is the universal large- $L$  limit of the quartic (Binder) cumulant at the critical point in the 2D Ising model. Since disorder is expected to be marginally irrelevant, see Sec. III A, the approach of  $\bar{U}_{22}$  and  $\bar{U}_4$  to their large- $L$  Ising limit is expected to be logarithmic.

The MC data of  $\bar{U}_4$  and  $\bar{U}_{22}$ , reported in Table I, clearly approach the Ising values (75). In the case of  $\bar{U}_4$ , see Table I,

TABLE I. MC data at fixed  $R_\xi=R_{\text{Is}}=0.905\,0488\,292(4)$ . For each model and lattice size  $L$ , we report the number of samples  $N_s$ , the quartic cumulants  $\bar{U}_4$  and  $\bar{U}_{22}$ , the magnetic susceptibility  $\chi$ , the derivative  $R'_\xi \equiv dR_\xi/d\beta$ , and the specific heat  $C_h$ . If the asymptotic behavior is controlled by the Ising fixed point, for  $L \rightarrow \infty$  we should have  $\bar{U}_4 \rightarrow U_{\text{Is}}=1.167\,923(5)$  and  $\bar{U}_{22} \rightarrow 0$ .

Model	$L$	$N_s/10^3$	$\bar{U}_4$	$\bar{U}_{22}$	$\chi$	$R'_\xi$	$C_h$
RSIM, $p=0.9$	8	5361	1.16476(3)	0.05083(3)	36.1853(9)	6.5911(11)	2.7285(5)
	16	2560	1.16463(4)	0.04170(4)	122.367(4)	12.608(5)	3.4282(12)
	32	1280	1.16507(4)	0.03618(4)	412.573(14)	24.029(9)	4.0283(14)
	64	640	1.16563(6)	0.03237(6)	1389.57(8)	45.91(3)	4.550(3)
	128	640	1.16597(6)	0.02947(6)	4677.0(3)	87.84(7)	5.014(3)
	256	653	1.16619(5)	0.02704(5)	15741.3(7)	168.50(9)	5.431(3)
	512	633	1.16656(4)	0.02522(5)	52962(2.0)	324.18(19)	5.815(3)
RSIM, $p=0.7$	8	640	1.14557(10)	0.09561(10)	25.841(3)	1.2536(13)	0.2155(3)
	16	2176	1.15206(6)	0.07526(6)	86.941(5)	2.6583(11)	0.30474(14)
	32	1280	1.15682(6)	0.06297(7)	293.888(15)	4.967(2)	0.35203(12)
	64	658	1.15996(7)	0.05491(8)	993.26(6)	9.140(4)	0.38283(10)
	128	843	1.16185(6)	0.04871(7)	3351.92(14)	16.903(6)	0.40516(7)
	256	1288	1.16313(4)	0.04368(5)	11299.1(3)	31.501(9)	0.42262(5)
	$\pm J$ Is, $p=0.95$	8	3200	1.16399(2)	0.04026(3)	40.9962(8)	6.0887(15)
16		3200	1.16405(3)	0.04023(3)	139.318(5)	11.163(2)	3.1527(8)
32		3200	1.16439(2)	0.03847(3)	470.511(11)	20.924(3)	3.6299(5)
64		812	1.16482(4)	0.03592(5)	1585.27(7)	39.570(10)	4.0371(7)
128		658	1.16527(4)	0.03331(6)	5337.0(3)	75.12(2)	4.3865(5)

the MC data are very close to  $U_{\text{Is}}=1.167\,923(5)$ . For the largest lattices, the relative difference  $\delta_4 \equiv |\bar{U}_4 - U_{\text{Is}}|/U_{\text{Is}}$  is very small:  $\delta_4 \approx 0.0012$ ,  $0.0041$ , and  $0.0023$  for the RSIM at  $p=0.9$  ( $L=512$ ) and  $p=0.7$  ( $L=256$ ), and the  $\pm J$  Ising model at  $p=0.95$  ( $L=128$ ), respectively. However, the asymptotic approach to the large- $L$  Ising value is very slow, hinting at logarithmic corrections. This is also strongly suggested by the MC data of  $\bar{U}_{22}$ , which are shown versus  $1/\ln L$  in Fig. 2.

In order to check the approach of the critical exponents to the Ising values, we define the effective exponents

$$\eta_{\text{eff}}(L) \equiv 2 - \frac{\ln \chi(2L)/\chi(L)}{\ln 2} \quad (76)$$

and

$$1/\nu_{\text{eff}}(L) \equiv \frac{\ln R'_\xi(2L)/R'_\xi(L)}{\ln 2}, \quad (77)$$

$$1/\nu_{U,\text{eff}}(L) \equiv \frac{\ln U'_4(2L)/U'_4(L)}{\ln 2}, \quad (78)$$

where we indicate the derivative with respect to  $\beta$  with a prime. The MC estimates of  $\eta_{\text{eff}}(L)$  and  $1/\nu_{\text{eff}}(L)$  are plotted in Figs. 3 and 4. They appear to approach the Ising values  $\eta = \frac{1}{4}$  and  $1/\nu = 1$ . In the case of  $\eta$ , the raw data are already

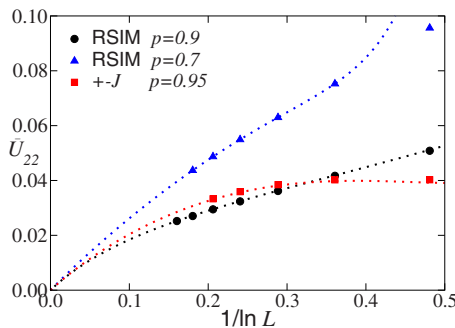


FIG. 2. (Color online) The phenomenological coupling  $\bar{U}_{22}$  vs  $1/\ln L$ . The lines show the results of fits to Eq. (87). For the RSIM at  $p=0.9$  and the  $\pm J$  Ising model we fit all data, while for the RSIM at  $p=0.7$  we use data satisfying  $L \geq 16$ . Note that the asymptotic slope as  $1/\ln L \rightarrow 0$  of the resulting curves is identical in the three cases, confirming the universality of  $C_{22,1}$ , defined by  $\bar{U}_{22} = C_{22,1} \bar{g}(\ln L) + O[\bar{g}(\ln L)^3]$ ; see Sec. IV B 3.

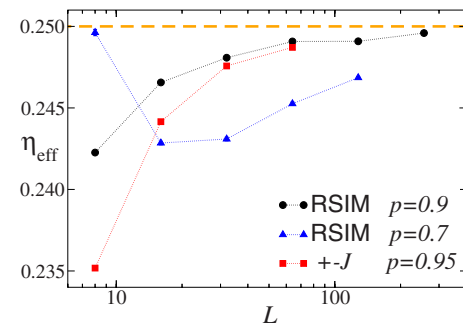


FIG. 3. (Color online) MC estimates of  $\eta_{\text{eff}}(L)$ . The dashed line corresponds to the Ising value  $\eta = \frac{1}{4}$ . The dotted lines are drawn to guide the eye.



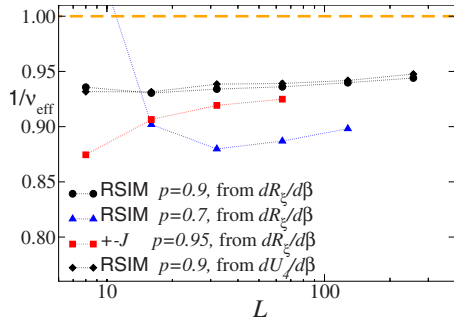


FIG. 4. (Color online) MC estimates of  $1/\nu_{\text{eff}}(L)$ . The dashed line corresponds to the Ising value  $1/\nu=1$ . The dotted lines are drawn to guide the eye.

very close to the Ising value: the largest lattices give  $\eta_{\text{eff}}(L=256)=0.249\,59(8)$  for the RSIM at  $p=0.9$ ,  $\eta_{\text{eff}}(L=128)=0.246\,86(8)$  for the RSIM at  $p=0.7$ , and  $\eta_{\text{eff}}(L=64)=0.248\,71(10)$  for the  $\pm J$  Ising model at  $p=0.95$ . In the case of  $1/\nu_{\text{eff}}(L)$ , the approach is much slower. At the largest available lattices, we find  $1/\nu_{\text{eff}}(L=256)=0.9441(12)$  for the RSIM at  $p=0.9$ ,  $1/\nu_{\text{eff}}(L=128)=0.8981(7)$  for the RSIM at  $p=0.7$ , and  $1/\nu_{\text{eff}}(L=64)=0.9249(5)$  for the  $\pm J$  Ising model at  $p=0.95$ . Anyway, all data show an upward trend toward the Ising value  $1/\nu=1$ .

These results provide quite strong evidence that the asymptotic behavior of the FSS is universal and is controlled by the Ising fixed point, with scaling corrections that decay very slowly. In the following, we report a more careful analysis of these logarithmic corrections, showing that they have an universal pattern that is consistent with the RG predictions obtained in Sec. III.

## 2. Universal finite-size behavior as a function of the phenomenological coupling $U_{22}$

As discussed in Sec. III B, the FSS formulas obtained from the RG equations of Sec. III A can be written in terms of the phenomenological coupling  $U_{22}$ . In the following, we compare the MC data with the predictions reported in Secs. III B and III C, and in particular with Eqs. (65)–(67).

Let us first consider the quartic cumulant  $\bar{U}_4$  defined in Eq. (14). At fixed  $R_\xi$ ,  $\bar{U}_4(L)$  is expected to behave as

$$\bar{U}_4(L) = \bar{f}_{U_4}(\bar{U}_{22}), \quad (79)$$

where  $\bar{f}_{U_4}(x)$  is a universal function, analytic at  $x=0$ , satisfying  $\bar{f}_{U_4}(0)=U_{\text{Is}}$ . Corrections to the behavior (79) vanish as powers of  $1/L$ . The scaling behavior (79) is well satisfied by the MC data, as shown in Fig. 5. All data fall on a single curve, except for a few of them corresponding to small values of  $L$  (this is particularly evident in the data for the RSIM at  $p=0.9$ ), indicating the presence of power-law scaling corrections. The results show that the linear term is absent or negligible in the expansion of  $\bar{f}_{U_4}(\bar{U}_{22})$  around  $\bar{U}_{22}=0$ ; if the data are plotted versus  $\bar{U}_{22}^2$ , they fall on an approximately straight line, suggesting that

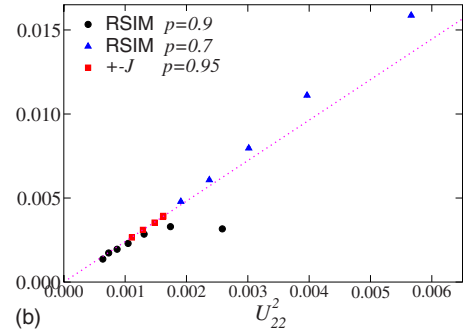
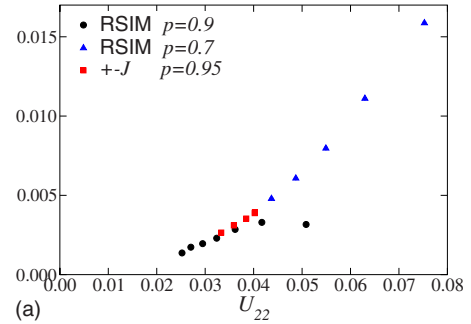


FIG. 5. (Color online)  $U_{\text{Is}} - \bar{U}_4$  vs  $\bar{U}_{22}$  (above) and  $\bar{U}_{22}^2$  (below).

$$\bar{U}_4(L) - U_{\text{Is}} = c\bar{U}_{22}(L)^2 + O(\bar{U}_{22}^3). \quad (80)$$

A fit of the numerical results to  $\bar{U}_4(L) - U_{\text{Is}} = c\bar{U}_{22}(L)^2$  gives  $c=2.4(2)$ . This implies that

$$\bar{U}_4(L) = U_{\text{Is}} + \frac{c_4}{(\ln L/L_0)^2} + \dots, \quad (81)$$

where  $L_0$  is the model-dependent constant that appears in the expansion of  $\bar{g}(\ln L)$  (as such, it is independent of the quantity that one is considering).

As discussed in Secs. III B and III C, at fixed  $R_\xi$ ,  $\chi$  behaves as

$$\chi = d_\chi L^{7/4} \bar{f}_\chi(\bar{U}_{22}(L)), \quad (82)$$

where  $\bar{f}_\chi(x)$  is a universal function such that  $\bar{f}_\chi(0)=1$ . This means that, by properly choosing constants  $e_\chi$ , the combination  $e_\chi \chi L^{-7/4}$  is a universal function of  $\bar{U}_{22}$ . In Fig. 6, we show this quantity. The plot is clearly consistent with Eq. (82). Note also that if the data are plotted versus  $\bar{U}_{22}^2$ , they approximately fall on a straight line, suggesting  $\bar{f}_\chi(x)=1 + O(x^2)$ , analogously to the case of  $\bar{U}_4$ .

In Fig. 4, we showed the effective exponents (77) and (78) related to the thermal exponent  $\nu$ . The data approached the Ising value  $\nu_{\text{Is}}=1$  with slowly decaying corrections. The effective exponents computed by using Eqs. (77) and (78) were very close, as shown in Fig. 4 for the RSIM at  $p=0.9$  (this is also true for the other model considered). For this reason, in the following we focus on  $R'_\xi$ . As discussed in Secs. III B and III C, the derivative  $R'_\xi$  at fixed  $R_\xi$  scales as

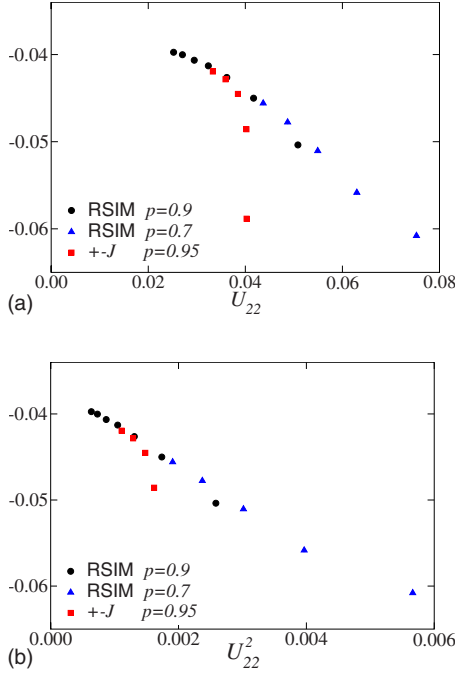


FIG. 6. (Color online) Plot of  $\ln(e_\chi \chi L^{-7/4})$  vs  $\bar{U}_{22}$  (top) and vs  $\bar{U}_{22}^2$  (bottom); we set  $e_\chi = 1, 1.4, 0.88$  for the RSIM at  $p=0.9$  and  $0.7$ , and for the  $\pm J$  Ising model. The constants  $e_\chi$  have been chosen so as to obtain the best collapse of the MC data.

$$R'_\xi = d_{dR} L \bar{U}_{22}(L)^{1/2} \bar{f}_{dR}(\bar{U}_{22}(L)), \quad (83)$$

where  $\bar{f}_{dR}(x)$  is a universal function. This means that, by properly choosing some constants  $e_{dR}$ , the combination  $e_{dR} R'_\xi / L$  is a universal function of  $\bar{U}_{22}$ . In Fig. 7, we show such a quantity. The plot is clearly consistent with Eq. (83): the data fall on a single curve and approach zero as  $\bar{U}_{22}(L)^{1/2}$  when  $\bar{U}_{22} \rightarrow 0$ . Again, note the presence of power-law corrections for large values of  $\bar{U}_{22}$ .

The approach of  $\beta_f(L)$  to  $\beta_c$  is given by Eq. (74). Equivalently, we can also consider the scaling form

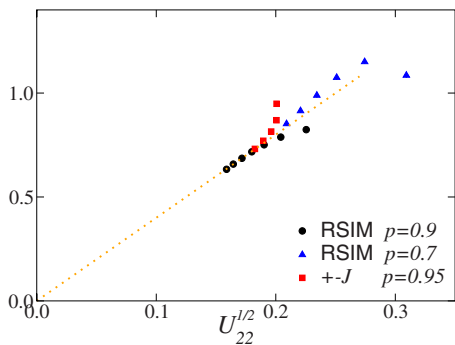


FIG. 7. (Color online) Plot of  $e_{dR} R'_\xi / L$  vs  $\bar{U}_{22}^{1/2}$ . We have chosen  $e_{dR} = 1, 6.9, 1.2$  for the RSIM at  $p=0.9$  and  $0.7$ , and for the  $\pm J$  Ising model. The constants  $e_{dR}$  have been chosen so as to obtain the best collapse of the MC data.

$$\beta_f(L) - \beta_c = \frac{d_1 \bar{U}_{22}(L)^{1/2}}{L} + \frac{d_2 \bar{U}_{22}(L)^{3/2}}{L} + \dots \quad (84)$$

We determine  $\beta_c$  by performing fits to Eq. (84). We include only data such that  $L \geq L_{\min}$ , where  $L_{\min}$  is the smallest cutoff that provides fits with  $\chi^2/\text{DOF} \leq 1$ . Moreover, as a check we have also performed fits to Eq. (84) in which we only consider the leading term (i.e., we set  $d_2=0$ ). For the RSIM at  $p=0.9$ , we obtain  $\beta_c = 0.525\,838(1)$  (fit with  $d_2=0$ ) and  $\beta_c = 0.525\,835(2)$ , if both terms are taken into account. Analogously, these two fits give  $\beta_c = 0.932\,94(1), 0.932\,89(3)$  for the RSIM at  $p=0.7$  and  $\beta_c = 0.533\,62(1), 0.533\,48(2)$  for the  $\pm J$  Ising model. Our final estimates are  $\beta_c = 0.525\,835(3), 0.932\,89(5)$ , and  $0.5335(1)$ , respectively, for the RSIM at  $p=0.9$  and  $0.7$ , and the  $\pm J$  Ising model at  $p=0.95$ . Consistent results are obtained by fitting the data of  $\beta_f(L)$  to Eq. (74).

### 3. Universal logarithmic corrections as a function of $L$

In the following, we directly check the dependence on  $L$  of  $\bar{U}_{22}$ ,  $R'_\xi$ , and of the specific heat  $C_h$ . As discussed in Sec. III C, for  $L \rightarrow \infty$  the phenomenological coupling  $\bar{U}_{22}$  behaves as

$$\bar{U}_{22}(L) = C_{22,1} \bar{g}(\ln L) + O(\bar{g}^3), \quad (85)$$

where  $C_{22,1}$  is a universal constant. The absence of the term of order  $\bar{g}^2$  fixes uniquely the normalization of the coupling  $\bar{g}$ . This quantity can be expanded in powers of  $1/\ln(L/L_0)$  to different orders. Using the expansion (34) with  $y = \ln L/L_0$ , we can perform three different types of fit, corresponding to three different approximations for  $\bar{g}(\ln L)$ . In fit (a), we fit  $\bar{U}_{22}(L)$  to

$$\bar{U}_{22}(L) = \frac{C_{22,1}}{\ln L/L_0}, \quad (86)$$

where  $C_{22,1}$  and  $L_0$  are free parameters. In fit (b), we also include the next term proportional to  $b_3$ , i.e., we fit the data to

$$\bar{U}_{22}(L) = \frac{C_{22,1}}{\ln L/L_0} - \frac{C_{22,1} b_3 \ln \ln L/L_0}{(\ln L/L_0)^2}, \quad (87)$$

where  $C_{22,1}$ ,  $b_3$ , and  $L_0$  are free parameters. Finally, we can also include the next term obtaining [fit (c)]

$$\bar{U}_{22}(L) = C_{22,1} \left\{ \frac{1}{\ln L/L_0} - \frac{b_3 \ln \ln L/L_0}{(\ln L/L_0)^2} + \frac{b_3^2 [(\ln \ln L/L_0)^2 - \ln \ln L/L_0]}{(\ln L/L_0)^3} \right\}, \quad (88)$$

where  $C_{22,1}$ ,  $b_3$ , and  $L_0$  are free parameters. The results of the fits for different values of  $L_{\min}$  are reported in Table II. Let us consider first the fit of the data for the RSIM at  $p=0.9$  for which we have the largest lattices. Fit (a) has a very poor  $\chi^2$ , indicating that the data are not well fitted by a single logarithmic term. If we include the next correction the  $\chi^2$  drops dramatically, indicating that our results are precise enough to be sensitive to the elusive  $\ln \ln L/L_0$  terms. Beside the very



for the RSIM at  $p=0.9$ . If we fit all data, we obtain  $a_1 = -0.53(5)$  ( $\chi^2/\text{DOF}=0.71$ ); if we discard the result corresponding to  $L=8$ , we obtain  $a_1 = -0.44(11)$ . These results are in good agreement with the theoretical prediction  $a_1 = -\frac{1}{2}$ .

Finally, we consider the specific heat  $C_h$ ,

$$C_h = \frac{[\langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2]}{V}, \quad (92)$$

where  $\mathcal{H}$  is the Hamiltonian. The RG analyses of Refs. [1,2,4,28] predict a diverging  $\ln \ln L$  asymptotic behavior. In Fig. 8, we show the MC data of  $C_h$  at fixed  $R_\xi = R_{Is}$ . They are definitely consistent with the theoretical prediction for its asymptotic behavior,

$$C_h \approx A \ln \ln(L/L_0) + B, \quad (93)$$

where  $A$  and  $B$  are nonuniversal parameters and  $L_0$  is the model-dependent scale. Fits of the data to Eq. (93), taking  $A$ ,  $B$ , and  $L_0$  as free parameters, suggest  $A \approx 5, 0.1, 2$  for the RSIM at  $p=0.9$  and  $0.7$ , and the  $\pm J$  Ising model at  $p=0.95$ , respectively. There is a large difference between the values for RSIM at  $p=0.9$  and  $0.7$ , but this should not be surprising

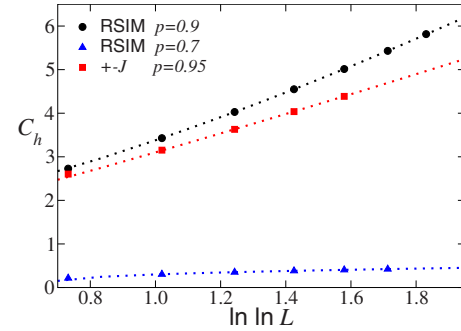


FIG. 8. (Color online) MC data of the specific heat versus  $\ln \ln L$ . The lines correspond to fits to Eq. (93).

because the  $p=0.7$  is quite close to the percolation point [46]  $p = p_{\text{perc}} \approx 0.59$ .

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